

03 JAN 14 (R)

1.

$$f(x) = \sec x + 3x - 2, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

(a) Show that there is a root of  $f(x) = 0$  in the interval  $[0.2, 0.4]$

(b) Show that the equation  $f(x) = 0$  can be written in the form

$$x = \frac{2}{3} - \frac{1}{3 \cos x}$$

The solution of  $f(x) = 0$  is  $\alpha$ , where  $\alpha = 0.3$  to 1 decimal place.

(c) Starting with  $x_0 = 0.3$ , use the iterative formula

$$x_{n+1} = \frac{2}{3} - \frac{1}{3 \cos x_n}$$

to calculate the values of  $x_1$ ,  $x_2$ , and  $x_3$ , giving your answers to 4 decimal places.

(d) State the value of  $\alpha$  correct to 3 decimal places.

a)  $f(0.2) = -0.38 < 0$   $\therefore$  by sign change rule  $\alpha \in [0.2, 0.4]$   
 $f(0.4) = 0.29 > 0$

b)  $\frac{1}{\cos x} + 3x - 2 = 0 \Rightarrow 3x = 2 - \frac{1}{\cos x}$

$\therefore x = \frac{2}{3} - \frac{1}{3 \cos x}$  #

c)  $x_0 = 0.3$  d)  $\alpha = 0.316$  (3dp)  
 $x_1 = 0.3177$   
 $x_2 = 0.3158$   
 $x_3 = 0.3160$

2.

$$f(x) = \frac{15}{3x+4} - \frac{2x}{x-1} + \frac{14}{(3x+4)(x-1)}, \quad x > 1$$

(a) Express  $f(x)$  as a single fraction in its simplest form.

(b) Hence, or otherwise, find  $f'(x)$ , giving your answer as a single fraction in its simplest form.

a) 
$$\frac{15(x-1) - 2x(3x+4) + 14}{(3x+4)(x-1)}$$

$$= \frac{15x - 15 - 6x^2 - 8x + 14}{(3x+4)(x-1)}$$

$$= \frac{-6x^2 + 7x - 1}{(3x+4)(x-1)}$$

$$= -\frac{(6x^2 - 7x + 1)}{(3x+4)(x-1)}$$

$$= -\frac{(6x-1)(x-1)}{(3x+4)(x-1)} = \frac{1-6x}{4+3x}$$

b)  $\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{vu' - uv'}{v^2}$   $u = 1-6x$   $v = 4+3x$   
 $u' = -6$   $v' = 3$

$$f'(x) = \frac{-6(4+3x) - 3(1-6x)}{(4+3x)^2} = \frac{-24 - 18x - 3 + 18x}{(4+3x)^2}$$

$$= \frac{-27}{(4+3x)^2}$$

3. (a) By writing  $\operatorname{cosec} x$  as  $\frac{1}{\sin x}$ , show that

$$\frac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x \cot x \quad (3)$$

Given that  $y = e^{3x} \operatorname{cosec} 2x$ ,  $0 < x < \frac{\pi}{2}$ ,

- (b) find an expression for  $\frac{dy}{dx}$ . (3)

The curve with equation  $y = e^{3x} \operatorname{cosec} 2x$ ,  $0 < x < \frac{\pi}{2}$ , has a single turning point.

- (c) Show that the  $x$ -coordinate of this turning point is at  $x = \frac{1}{2} \arctan k$  where the value of the constant  $k$  should be found. (2)

$$\begin{aligned} a) \frac{d}{dx} \operatorname{cosec} x &= \frac{d}{dx} (\sin x)^{-1} \\ &= -(\sin x)^{-2} \times \cos x \\ &= -\frac{1}{\sin x} \times \frac{\cos x}{\sin x} = -\operatorname{cosec} x \cot x \end{aligned}$$

$$\begin{aligned} b) y &= e^{3x} \operatorname{cosec} 2x \\ u &= e^{3x} \quad v = \operatorname{cosec} 2x \\ u' &= 3e^{3x} \quad v' = -2 \operatorname{cosec} 2x \cot 2x \end{aligned}$$

$$\begin{aligned} y' &= vu' + uv' \\ y' &= 3e^{3x} \operatorname{cosec} 2x - 2e^{3x} \operatorname{cosec} 2x \cot 2x \\ &= e^{3x} \operatorname{cosec} 2x (3 - 2 \cot 2x) \end{aligned}$$

- c) TP when  $y' = 0$

$$\begin{aligned} e^{3x} \operatorname{cosec} 2x &= 0 & 3 - 2 \cot 2x &= 0 \\ \text{No solutions} & & 2 \cot 2x &= 3 \\ & & \cot 2x &= \frac{3}{2} \\ & & \Rightarrow \tan 2x &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \Rightarrow 2x &= \arctan\left(\frac{2}{3}\right) \\ \therefore x &= \frac{1}{2} \arctan\left(\frac{2}{3}\right) \end{aligned}$$

4. A pot of coffee is delivered to a meeting room at 11am. At a time  $t$  minutes after 11am the temperature,  $\theta^\circ\text{C}$ , of the coffee in the pot is given by the equation

$$\theta = A + 60e^{-kt}$$

where  $A$  and  $k$  are positive constants.

Given also that the temperature of the coffee at 11am is  $85^\circ\text{C}$  and that 15 minutes later it is  $58^\circ\text{C}$ ,

- (a) find the value of  $A$ . (1)
- (b) Show that  $k = \frac{1}{15} \ln\left(\frac{20}{11}\right)$ . (3)
- (c) Find, to the nearest minute, the time at which the temperature of the coffee reaches  $50^\circ\text{C}$ . (4)

$$\theta = A + 60e^{-kt} \quad t=0, \theta=85 \quad \therefore A=25$$

$$\begin{aligned} b) t=15 \quad 58 &= 25 + 60e^{-15k} \\ \theta=58 &\Rightarrow 33 = 60e^{-15k} \Rightarrow e^{-15k} = \frac{11}{20} \\ \Rightarrow -15k &= \ln\left(\frac{11}{20}\right) \Rightarrow -15k = \ln\left(\frac{20}{11}\right)^{-1} \\ \Rightarrow -15k &= -\ln\left(\frac{20}{11}\right) \therefore k = \frac{1}{15} \ln\left(\frac{20}{11}\right) \end{aligned}$$

$$\begin{aligned} c) 50 &= 25 + 60e^{-\frac{1}{15} \ln\left(\frac{20}{11}\right)t} \\ \Rightarrow e^{-\frac{1}{15} \ln\left(\frac{20}{11}\right)t} &= \frac{5}{12} \Rightarrow -\frac{1}{15} \ln\left(\frac{20}{11}\right)t = \ln\left(\frac{5}{12}\right) \\ \Rightarrow \frac{1}{15} \ln\left(\frac{20}{11}\right)t &= \ln\left(\frac{12}{5}\right) \therefore t = \frac{15 \ln\left(\frac{12}{5}\right)}{\ln\left(\frac{20}{11}\right)} \\ \therefore t &= 21.96 \approx \underline{22 \text{ min}} \end{aligned}$$

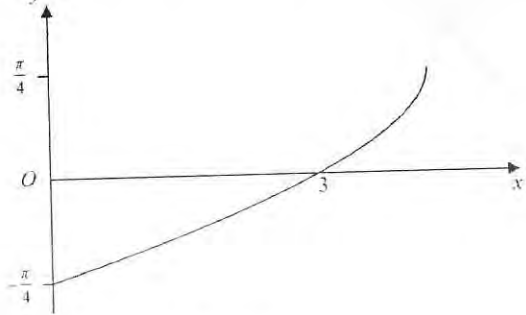


Figure 1

The curve shown in Figure 1 has equation

$$x = 3 \sin y + 3 \cos y, \quad -\frac{\pi}{4} < y < \frac{\pi}{4}$$

(a) Express the equation of the curve in the form

$$x = R \sin(y + \alpha), \text{ where } R \text{ and } \alpha \text{ are constants, } R > 0 \text{ and } 0 < \alpha < \frac{\pi}{2} \quad (3)$$

(b) Find the coordinates of the point on the curve where the value of  $\frac{dy}{dx}$  is  $\frac{1}{2}$ .

Give your answers to 3 decimal places. (6)

$$R \sin(y + \alpha) = R \sin y \cos \alpha + R \cos y \sin \alpha$$

$$x = 3 \sin y + 3 \cos y$$

$$\frac{R \sin \alpha}{R \cos \alpha} = \frac{3}{3} \quad \tan \alpha = 1 \quad \therefore \alpha = \frac{\pi}{4}$$

$$R = \sqrt{18} = 3\sqrt{2}$$

$$\therefore x = 3\sqrt{2} \sin\left(y + \frac{\pi}{4}\right)$$

$$\frac{dx}{dy} = 3\sqrt{2} \cos\left(y + \frac{\pi}{4}\right) = 2 \quad \left(\frac{dy}{dx} = \frac{1}{2}\right)$$

$$\therefore \cos\left(y + \frac{\pi}{4}\right) = \frac{2}{3\sqrt{2}} \quad y + \frac{\pi}{4} = 1.0799 \quad \therefore y = 0.295$$

$$x = 3\sqrt{2} \left(\sin\left(y + \frac{\pi}{4}\right)\right) \quad \therefore x = 3.742$$

$$(3.742, 0.295)$$

6. Given that  $a$  and  $b$  are constants and that  $0 < a < b$ ,

(a) on separate diagrams, sketch the graph with equation

(i)  $y = |2x + a|$ .

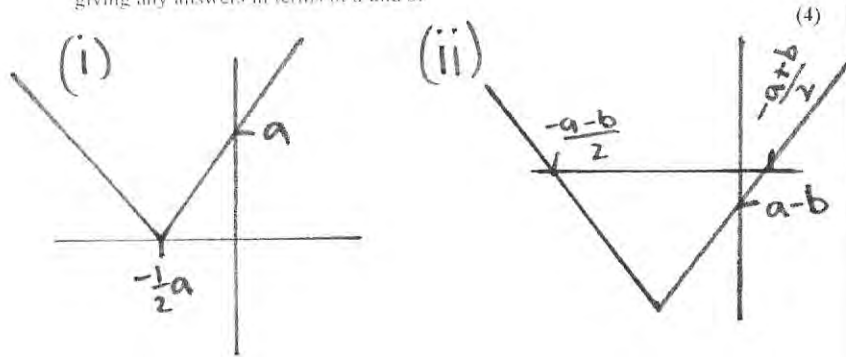
(ii)  $y = |2x + a| - b$ .

Show on each sketch the coordinates of each point at which the graph crosses or meets the axes. (6)

(b) Solve, for  $x$ , the equation

$$|2x + a| - b = \frac{1}{3}x$$

giving any answers in terms of  $a$  and  $b$ .



b)  $|2x + a| = \frac{1}{3}x + b$

$$2x + a = \frac{1}{3}x + b$$

$$\frac{5}{3}x = b - a$$

$$x = \frac{3b - 3a}{5}$$

$$-2x - a = \frac{1}{3}x + b$$

$$-a - b = \frac{1}{3}x$$

$$x = \frac{-3a - 3b}{7}$$

7. (i) (a) Prove that

$$\cos 3\theta \equiv 4 \cos^3 \theta - 3 \cos \theta$$

(You may use the double angle formulae and the identity  $\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$ )

(4)

(b) Hence solve the equation

$$2 \cos 3\theta + \cos 2\theta + 1 = 0$$

giving answers in the interval  $0 \leq \theta \leq \pi$ .

Solutions based entirely on graphical or numerical methods are not acceptable.

(6)

(ii) Given that  $\theta = \arcsin x$  and that  $0 < \theta < \frac{\pi}{2}$ , show that

$$\cot \theta = \frac{\sqrt{1-x^2}}{x}, \quad 0 < x < 1$$

(3)

$$\begin{aligned} \text{a) } \cos 3\theta &= \cos(2\theta + \theta) = \cos 2\theta \cos \theta - \sin 2\theta \sin \theta \\ &= (2\cos^2 \theta - 1)\cos \theta - 2\sin^2 \theta \sin \theta \\ &= 2\cos^3 \theta - \cos \theta - 2\cos \theta(1 - \cos^2 \theta) \\ &= 2\cos^3 \theta - \cos \theta - 2\cos \theta + 2\cos^3 \theta \\ &= 4\cos^3 \theta - 3\cos \theta \quad \# \end{aligned}$$

$$\begin{aligned} \text{b) } 8\cos^3 \theta - 6\cos \theta + \cos 2\theta + 1 &= 0 \\ 8\cos^3 \theta - 6\cos \theta + 2\cos^2 \theta - 1 + 1 &= 0 \\ = 2\cos \theta (4\cos^2 \theta + \cos \theta - 3) &= 0 \\ = 2\cos \theta (4\cos \theta - 3)(\cos \theta + 1) &= 0 \end{aligned}$$

$$\therefore \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

$$\therefore \cos \theta = \frac{3}{4} \Rightarrow \theta = 0.723^\circ$$

$$\therefore \cos \theta = -1 \Rightarrow \theta = \pi$$

$$\text{ii) } \theta = \arcsin x \Rightarrow \sin \theta = x$$

$$\begin{aligned} \sin^2 \theta = x^2 \Rightarrow 1 - \sin^2 \theta = 1 - x^2 &\Rightarrow \cos^2 \theta = 1 - x^2 \\ \Rightarrow \cos \theta = \sqrt{1 - x^2} \end{aligned}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{\sqrt{1-x^2}}{x} \quad \#$$

8. The function  $f$  is defined by

$$f: x \rightarrow 3 - 2e^{-x}, \quad x \in \mathbb{R}$$

(a) Find the inverse function,  $f^{-1}(x)$  and give its domain.

(5)

(b) Solve the equation  $f^{-1}(x) = \ln x$ .

(4)

The equation  $f(t) = ke^t$ , where  $k$  is a positive constant, has exactly one real solution.

(c) Find the value of  $k$ .

(4)

$$\begin{aligned} \text{a) } y &= 3 - 2e^{-x} \Rightarrow x = 3 - 2e^{-y} \\ \Rightarrow 2e^{-y} &= 3 - x \Rightarrow e^{-y} = \frac{3-x}{2} \\ \Rightarrow -y &= \ln \left| \frac{3-x}{2} \right| \Rightarrow y = \ln \left| \frac{2}{3-x} \right| \quad \underline{x < 3} \end{aligned}$$

$$\begin{aligned} \text{b) } \ln \left| \frac{2}{3-x} \right| &= \ln x \Rightarrow \frac{2}{3-x} = x \Rightarrow 2 = 3x - x^2 \\ x^2 - 3x + 2 &= 0 \quad (x-2)(x-1) = 0 \\ \Rightarrow \underline{x=2}, \underline{x=1} \end{aligned}$$

$$\begin{aligned} \text{c) } f(t) &= 3 - 2e^{-t} = ke^t \\ 3 - \frac{2}{e^t} &= ke^t \quad (xe^t) \quad 3e^t - 2 = ke^{2t} \\ ke^{2t} - 3e^t + 2 &= 0 \end{aligned}$$

$$\text{one solution} \Rightarrow b^2 - 4ac = 0 \quad 9 - 8k = 0 \quad k = \frac{9}{8}$$